Structural Model Equivalence

James H. Steiger

Department of Psychology and Human Development
Vanderbilt University

Structural Model Equivalence

Introduction

- In the very early days of structural equation modeling, the software options were few.
- When I taught my first course around 1980, LISREL and COSAN were available on mainframes, and you submitted your jobs on decks of computer cards, and waited at least an hour for the output.
- So a round of submitting a model and getting all the errors out of your model specification could take days, instead of minutes.
- The unwieldy nature of the process made certain things very difficult to see, and discouraged "exploratory fiddling" with a model specification.
- Changing the direction of one arrow might result in a two day excursion into fixing up your LISREL input. Since, in those days, we were actually paying for computer cycles used, a failure to converge on a single large LISREL model could cost several hundred dollars!

Introduction

- By 1983, the IBM PC was a reality, and its 256K of memory made structural equation modeling feasible on a microcomputer. By 1985, several of us had "user-friendly" systems in the works, which enabled us to re-specify models and fit them automatically, with sharply reduced likelihood of specification error.
- A few years later, several "user-friendly" programs were in circulation, which allowed models to be specified in a "path diagram-like" syntax. I had just noticed a 1987 *Child Development* paper written by some colleague-friends, George Huba and Lisa Harlow, and, as a homework assignment, asked my class to reproduce their analysis.
- The paper involved the causal sequence of drug use in teenagers, and assessed whether beer drinking had an impact on marijuana use.
- Some students came to me and told me that my program, EzPATH, must have an error in it, because they kept getting the same chi-square statistic for different models.

Introduction

- On investigation, we found that, indeed, there was no error in the software. Different models, with different implications, fit the data equally well. In this case, we say that the models are *empirically equivalent*.
- It turned out that research in this area was just beginning. Stelzl (1986) had a paper in *Multivariate Behavioral Research* that was somewhat difficult for practitioners to follow.
- Indeed, despite the fact that, as Editor of *MBR* at the time, I had read the paper, I didn't see its application to the Huba-Harlow paper right away.
- In 1990, Lee and Hershberger published a very influential paper (also in *Multivariate Behavioral Research*) which delivered on the promise of its title, "A Simple Rule for Generating Equivalent Models in Covariance Structure Modeling."
- Lee and Hershberger (1990) concentrated on replacement rules for structural models. Consequently, their results are primarily applicable to the "structural part" of a covariance structure model, or for path models with manifest variables.
- In a subsequent article, Hershberger (2006) discussed equivalent measurement models, a topic we'll address in a companion lecture module.
- Let's quickly discuss the main ideas in the Lee-Hershberger 1990 paper and see how it applies to the Huba-Harlow research. Before describing their result, we need to develop some simple terminology.

Some Basic Terminology

- In this section, we develop the basic glossary terms necessary to efficiently follow the discussion in the Hershberger-Lee article.

Some Basic Terminology

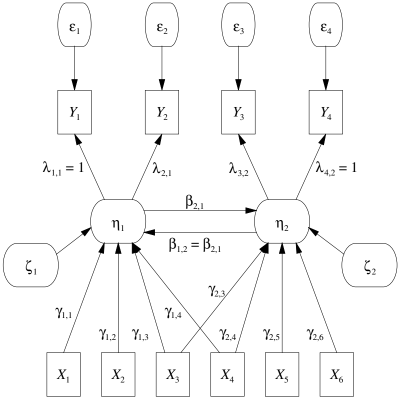
Recursive and Non-Recursive Models

- A path diagram describes a recursive model if the following conditions hold:
  1. All causal effects in the model are unidirectional, i.e. no two variables in the model are reciprocally related, either directly or indirectly. Hence, you can order the variables left to right so that the first endogenous variable is affected only by the exogenous variables. The 2nd endogenous variable is affected only by the exogenous variables and the first endogenous variable; and so on.
  2. All pairs of error (or disturbance) terms in the model are assumed to be uncorrelated.
- If a model is not recursive, then it is said to be non-recursive.

Some Basic Terminology

Recursive and Non-Recursive Models

Is the model in the diagram below recursive, or non-recursive?



Some Basic Terminology

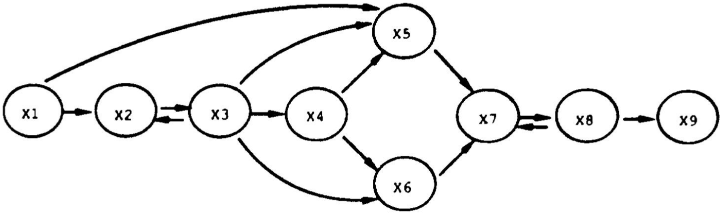
Preceding, Focal, and Succeeding Blocks in a Model

- Consider the structural portion of a path diagram. We can break the structural model into 3 sections, by surrounding each section with a box. If we do that,
  1. The middle section is called the *focal block* (FBL), and, assuming the diagram is organized so that causal flow is from left to right,
  2. The left block is called the *preceding block* (PBL),
  3. The right block is called the *succeeding block* (SBL)

Some Basic Terminology

Preceding, Focal, and Succeeding Blocks in a Model

- We can define the PBL, FBL, and SBL in a model by simply naming the variables in each block.
- Here is an example.
- The diagram below shows a structural model shown in Figure 1a of Lee and Hershberger (1990). We can break this model into a PBL (X1,X2,X3), a FBL (X4,X6), and an SBL (X5,X7,X8,X9).



Model 1A

Some Basic Terminology

Block-Recursive Model

- A structural model is block-recursive if, across blocks, the flow is recursive.
- It is important to emphasize that, within a block, the model can be either recursive or non-recursive.

Some Basic Terminology

Limited Block-Recursive Model

- A model is Limited Block-Recursive if it is block-recursive, *and* the FBL is also recursive.
- The PBL and SBL may be either recursive or non-recursive.

Some Basic Terminology

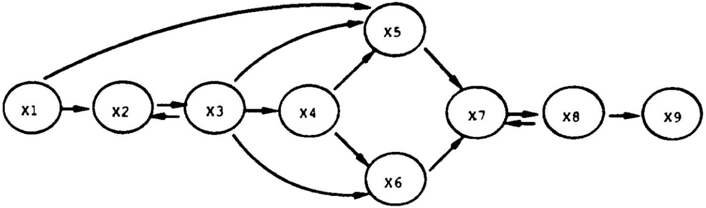
Symmetric Focal Block

- A focal block is symmetric if variables in it have the same predictors in the PBL.

Some Basic Terminology

Symmetric Focal Block

- Consider the diagram below with, again, a breakdown into a PBL (X1,X2,X3), a FBL (X4,X6), and an SBL (X5,X7,X8,X9). Answer the following questions:
  1. Is the PBL recursive or non-recursive?
  2. Is the SBL recursive or non-recursive?
  3. Is the FBL recursive or non-recursive?
  4. Is the model as broken down block-recursive? Limited block-recursive?
  5. Is the FBL symmetric?



Model 1A

The Lee-Hershberger Replacement Rules

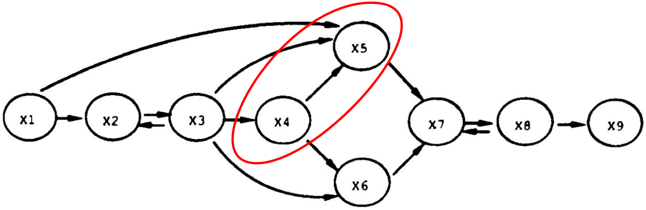
Basic Definition

- Suppose we break a structural model into a PBL,FBL, and SBL, and the model is limited block-recursive.
- The following hold:
  1. A direct path $X \leftarrow Y$ in the FBL may be replaced by a residual correlation between them.
  2. Moreover, any residual correlation between Ux and Uy within a FBL where the condition of limited block recursiveness holds can be replaced by a direct path, $X \leftarrow Y$ or $Y \leftarrow X$.
  3. The choice requires that, after the change, the endogenous variable has the same or more predictors in the PBL as the exogenous variable.
- Moreover, these rules can operate in reverse:
  1. Any residual correlation within a focal block where the condition of limited block-recursiveness holds in the model can be replaced by a direct path, $X \leftarrow Y$ or $X \rightarrow Y$. The choice requires that the effect variable has the same or more predictors in the preceding block (PBL) than the source variable has after the change.
- In the special case where the FBL is *symmetric*:
  1. Reversing path direction,
  2. Replacing the direct path with a residual covariance, or
  3. Replacing a residual covariance with a direct path of arbitrary direction between the symmetric variables generates a model equivalent to the original model.

The Lee-Hershberger Replacement Rules

An Example

- Consider Model 1A again. Treat $(X4, X5)$ as a FBL, $(X1, X2, X3)$ as a PBL, and $(X6, X7, X8, X9)$ as the SBL.
- The model is shown below with the FBL lassoed in red.



Model 1A

- In that case, we have limited block recursiveness.
- With limited block recursiveness, we can replace the path from $X4$ to $X5$ by putting residuals on them, then adding a covariance path between the residuals.

The Lee-Hershberger Replacement Rules

Special Treatment for Just-Identified Blocks

- When all parameters in a block can always be chosen so as to perfectly reproduce the variances and covariances of the variables within the block (and those parameters are unconstrained relative to other parameters in the model), the block is said to be just-identified (JID).
- Suppose a PBL is just-identified.
- Because the identification and estimation of a PBL can be dealt with independently of the succeeding blocks in a block-recursive system, a PBL can be viewed as a model itself.

The Lee-Hershberger Replacement Rules

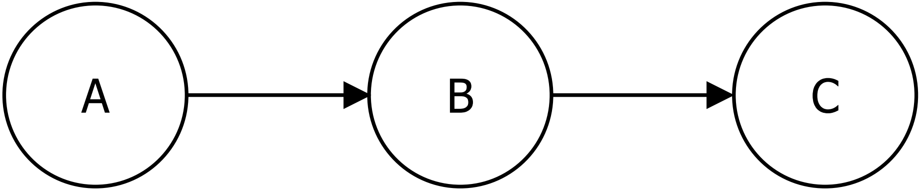
Special Treatment for Just-Identified Blocks

- Then a just-identified PBL can be considered as a just-identified model, in which case any other just-identified model for the variables in that PBL should generate an equivalent version of the larger model in which the PBL is embedded.
- The typical case of a just-identified PBL occurs when any two variables are connected by either a direct path or residual correlation, but not by both.
- Note that, when equivalent models are generated by applying the replacing rule to a JID block, the JID block can be considered a focal block.
- Because all variables can be completely connected by only covariances and all correlated variables are symmetric in a JID block, we can assume that all the variables in a JID block are symmetric variables.
- Consequently, changing the path direction and replacing a (residual) correlation with a direct path of arbitrary direction or replacing a direct path with (residual) correlation in a JID block can be considered just another way of applying the replacing rule on symmetric variables.

The Lee-Hershberger Replacement Rules

Special Treatment for Just-Identified Blocks

- Lee and Hershberger emphasize the *repeated* application of the replacing rule to the same diagram.
- In some cases, once a replacement rule is applied the first time, it opens up additional possibilities for applying the replacement rules again. For example, consider the very simple model shown below.

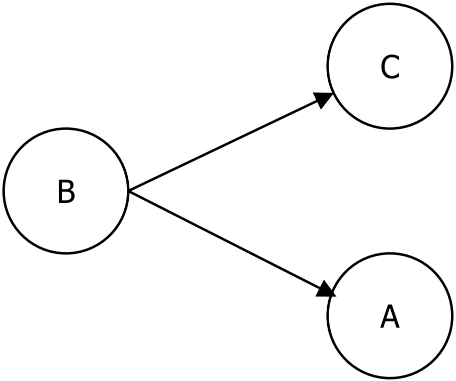


- Can you identify a JID PBL in this model?

The Lee-Hershberger Replacement Rules

Special Treatment for Just-Identified Blocks

- Great! Since the (A, B) block is JID, we can reverse the path between A and B, and redraw the model below.

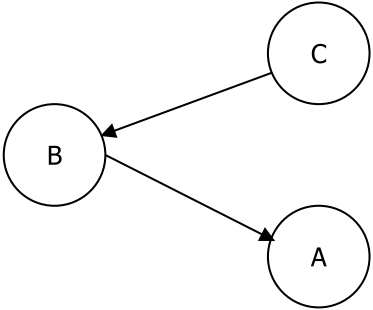


- Can you spot a JID FBL in this new model?
- If so, you can reverse the path between the two variables in this FBL.

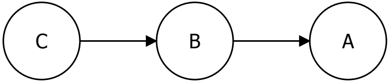
The Lee-Hershberger Replacement Rules

Special Treatment for Just-Identified Blocks

- After reversing the path, you get a new model, two versions of which appear below.



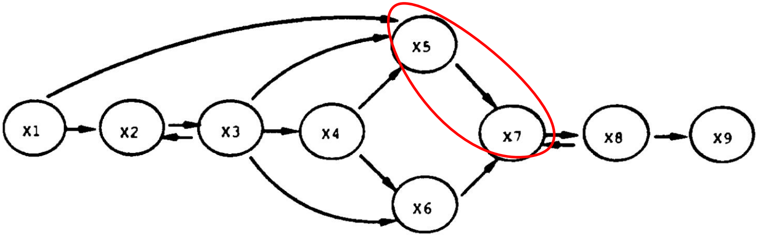
- The second version makes it clearer that even though the original model was not JID, we can still do a complete path reversal.



- Notice that the replacing rule, as written, was not applicable to the (B, C) block in the original model.

Some Example Applications

- Now we'll try some more complex examples.
- Consider Model IA from Lee and Hershberger (1990) again, but this time imagine the FBL $(X5, X7)$, with PBL $(X1, X2, X3, X4, X6)$ and SBL $(X8, X9)$, as shown below.

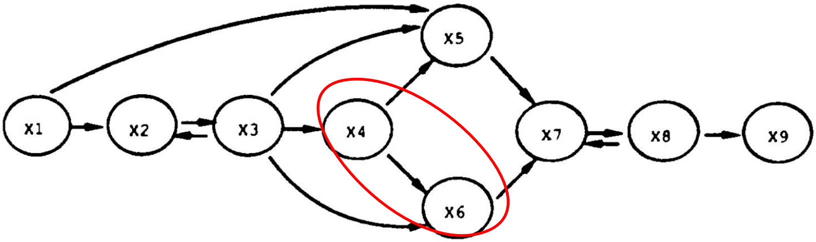


Model 1A

- Is this model block-recursive?
- Can the path from X5 to X7 be replaced by a correlated residuals on X5 and X7?

Some Example Applications

- Now, consider Model IA again, but now with $(X4, X6)$ as the FBL, as shown below.

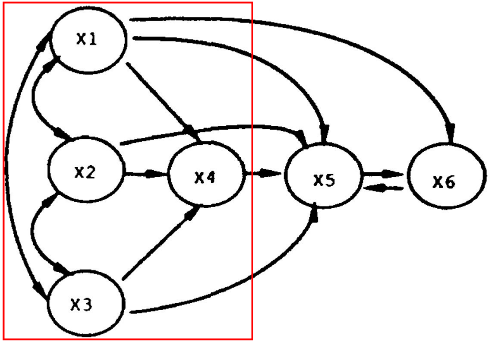


Model 1A

- Is this model block-recursive?
- Is this model limited block-recursive?

Some Example Applications

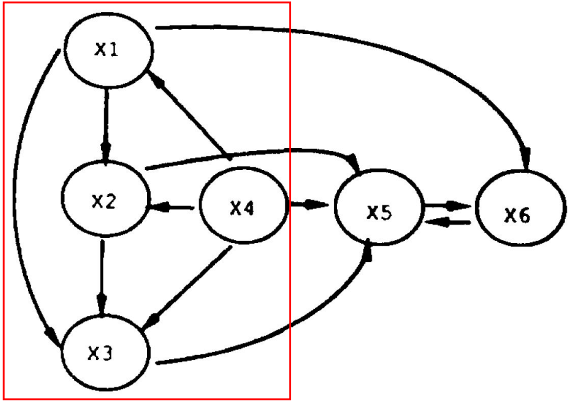
- Next, consider another model presented by Lee and Hershberger (1990).
- Suppose we isolate variables $(X1, X2, X3, X4)$, and consider this block to be PBL.
- What paths are implicit for the variables in this PBL?
- Is this block JID?



Model 1B

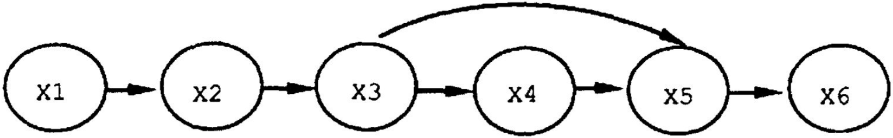
Some Example Applications

- Notice that, because the PBL is JID, we can reverse paths.
- Reversing several paths, we now end up with a rather different model, as shown below.



Some Example Applications

- Here is another practice example from Lee and Hershberger (1990).
- Can you spot a JID PBL and a symmetric focal block that is also JID?
- What path reversals can you achieve?
- Is your modified model recursive?

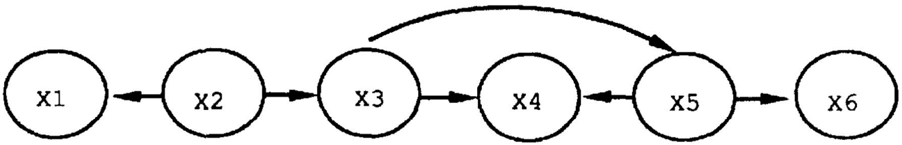


Model 4A

Some Example Applications

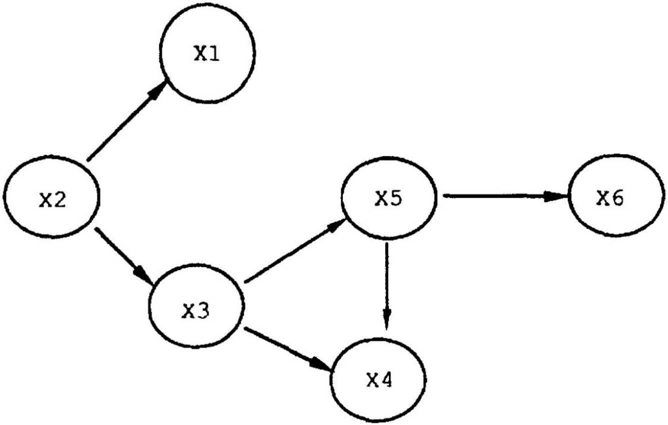
- On the next two slides, we see two drawings of the same model.
- It is somewhat easier (form most people) to evaluate recursiveness in the second drawing.

Some Example Applications



Model 4B

Some Example Applications



Model 4C

Proof of the Replacing Rule

- Lee and Hershberger give a proof of their replacing rule, a slightly modified version of which appears below
- Consider a $X \rightarrow Y$ relationship within a FBL.
- Let X have direct paths from predictors $(P_1, \ldots, P_m)$ in the PBL, and let Y have paths from both $(P_1, \ldots, P_m)$ and $(Q_1, \ldots, Q_n)$ in the PBL.
- We can describe the structural equations for X and Y generically as

$$X = F(P_1, \ldots, P_m) + u \quad (1)$$

$$Y = G(P_1, \ldots, P_m) + H(Q_1, \ldots, Q_n) + bX + v \quad (2)$$

where u and v are uncorrelated residual variables for X and Y, respectively. (There is a typo in Lee and Hershberger, where u and v are said to have a "non-zero" covariance.)
- b,F,G, and H represent path coefficients.

Proof of the Replacing Rule

- Suppose we substitute Equation **??** into Equation 2, obtaining, after a bit of rearrangement,

$$Y = G(P_1, \ldots, P_m) + H(Q_1, \ldots, Q_n) + bF(P_1, \ldots, P_m) + bu + v \tag{3}$$

which can be written in a simplified notation as

$$Y = J(P_1, \ldots, P_m) + H(Q_1, \ldots, Q_n) + e \tag{4}$$

where $e = bU + v$ and $J(P_1, \ldots, P_m) = bF(P_1, \ldots, P_m) + G(P_1, \ldots, P_m)$.

- We see that Y can be "considered" not to have a path from X in this reparamaterization.

Proof of the Replacing Rule

- However, what about the covariance between v and the redefined residual e? Clearly, by the algebra of variances and covarariances, it is

$$\text{Cov}(u, e) = \text{Cov}(u, bu + v) = b\,\text{Var}(u) + Cov(u, v) = b\,\text{Var}(u) + 0 = b\,\text{Var}(u) \qquad (5)$$

- Hence, deleting the path from X to Y retains the same model form, except that now the residual covariance becomes non-zero, and so that residual covariance path must be added to the model.
- The above proof implies that in a similar situation, the covariance path between X and Y can be replaced by a path between them, so long as the predictor set for the exdogenous variable in the pair is a subset of the predictor set for the endogenous variable in the pair.

An Application of the Replacing Rule

- On the next slide is the article by Huba and Harlow (1987).
- Can you find a symmetric focal block and reverse a key path in the diagram?

An Application of the Replacing Rule